# Combinatorial interpretation of Haldane-Wu fractional exclusion statistics

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(Received 16 February 2002; published 26 August 2002)

Assuming that the maximal allowed number of identical particles in a state is an integer parameter, $q$, we derive the statistical weight and analyze the associated equation that defines the statistical distribution. The derived distribution covers Fermi-Dirac and Bose-Einstein ones in the particular cases $q=1$ and $q\to\infty$ $(n_i/q\to 1)$, respectively. We show that the derived statistical weight provides a natural combinatorial interpretation of Haldane-Wu fractional exclusion statistics, and present exact solutions of the distribution equation.

DOI: 10.1103/PhysRevE.66.026116 PACS number(s): 05.30.Pr

## I. INTRODUCTION

Statistics that are different from Fermi-Dirac and Bose-Einstein ones become of much interest in various aspects. A recent example is given by Haldane-Wu fractional exclusion statistics [1,2], which is used to describe elementary excitations of a number of exactly solvable one-dimensional models of strongly correlated systems, and other models [2,3]. This statistics is based on the statistical weight, which is a generalization of Yang-Yang [4] state counting as mentioned by Wu,

$$W_i=\frac{[z_i+(n_i-1)(1-\lambda)]!}{n_i![z_i-\lambda n_i-(1-\lambda)]!}, \tag{1}$$

where the parameter $\lambda$ varies from $\lambda=0$ (Bose Einstein) to $\lambda=1$ (Fermi Dirac). This formula is a simple generalization and interpolation of Fermi and Bose statistical weights. While there is no physical meaning ascribed to $\lambda$ here, the physical interpretation of Eq. (1) is that the *effective* number of available single-particle states *linearly* depends on the number of particles,

$$z_i^f=z_i-(1-\lambda)(n_i-1), \quad z_i^b=z_i-\lambda(n_i-1) \tag{2}$$

for fermions and bosons, respectively. This is viewed as a defining feature of the fractional exclusion statistics.

In the present paper, we show that the equation that defines Haldane-Wu statistical distribution can be derived from a different statistical weight, which has a clear combinatorial and physical treatment. Also, we present exact solutions of this equation.

## II. THE COMBINATORICS

A number of quantum states of $n_i$ identical particles occupying $z_i$ states, with up to $q$ particles in state, $1\leqslant q\leqslant n_i$, can be counted as follows.

We consider a configuration defined as that it has a maximal possible number of totally occupied states (exactly $q$ particles in state). A number of such totally occupied states is an integer part of $n_i/q$ that we denote by $[n_i/q]$. If $q$ is a divisor of $n_i$ we have identically $[n_i/q]=n_i/q$, so that the number of unoccupied states is $z_i-(n_i/q)$. If $q$ is not a divisor of $n_i$ we have one partially occupied state, so that the number of unoccupied states is $z_i-(n_i/q)-1$. We write a combined formula of the statistical weight for both the cases as

$$W_i=\frac{\left(z_i+n_i-\left[\frac{n_i}{q}\right]\right)!}{n_i!\left(z_i-\left[\frac{n_i}{q}\right]-l\right)!}, \tag{3}$$

where $l=0$ or 1 if $n_i/q$ is an integer or a noninteger, respectively; $i=1,2,\dots,m$.

In these particular cases, $q=1$ and $q=n_i$, we have $[n_i/q]=n_i/q$ and $l=0$ so that Eq. (3) reduces to Fermi-Dirac and Bose-Einstein statistical weights, respectively,

$$W_i=\frac{z_i!}{n_i!(z_i-n_i)!}, \quad W_i=\frac{(z_i+n_i-1)!}{n!(z_i-1)!}. \tag{4}$$

As one can see, the effective number of available single-particle states derived from Eq. (3),

$$z_i^f=z_i-n_i+\left[\frac{n_i}{q}\right], \quad z_i^b=z_i-\left[\frac{n_i}{q}\right]+1 \tag{5}$$

for fermions and bosons, respectively, is linear in $n_i$ for integer $n_i/q$. With the identification of the parameters, $1/q=\lambda$, and the redefinition, $z_i\to z_i-(1-\lambda)$, the statistical weight (3) coincides with Haldane-Wu statistical weight (1), for the case of integer $n_i/q$. Consequently, the obtained statistical weight (3) corresponds to a kind of fractional exclusion statistics. To verify whether Eq. (3) leads to Haldane-Wu distribution we obtain below the equation that governs statistical distribution.

## III. THE DISTRIBUTION FUNCTION

Starting with Eq. (3), we follow usual technique of statistical mechanics to derive the associated most-probable distribution of $n_i$.

The thermodynamical probability is $W=\Pi W_i$, and the entropy, $S=k\ln W$, can be calculated by using the approximation of big number of particles, $n!\simeq n^n e^{-n}$ for big $n$. Assuming conservation of the total number of particles, $N=\Sigma n_i$ and the total energy, $E=\Sigma n_i\varepsilon_i$, variational study of $S$ corresponding to an equilibrium state gives us

 

$$\delta S = k\sum_i \left[\left(1-\frac{1}{q}\right)\ln\left(n_i+z_i-\frac{n_i}{q}\right)-\ln n_i + \frac{1}{q}\ln\left(z_i-\frac{n_i}{q}\right)-\alpha-\beta\varepsilon_i\right]\delta n_i=0, \tag{6}$$

where $\alpha$ and $\beta$ are Lagrange multipliers, and we have used $[n_i/q]\simeq n_i/q$ and $l=0$ for big $n_i$. Using the notation $\kappa=1/q$ and inserting $\alpha=-\mu/kT$ and $\beta=1/kT$ (obtained via an identification of $S$, at $q=1$, with the thermodynamical expression), we rewrite Eq. (6) as

$$\frac{[z_i+(1-\kappa)n_i]^{1-\kappa}(z_i-\kappa n_i)^{\kappa}}{n_i}=\exp\frac{\varepsilon_i-\mu}{kT}, \tag{7}$$

$\kappa=1,\frac{1}{2},\frac{1}{3},\ldots$. To draw parallels with Haldane-Wu statistics below we make analytic continuation of the discrete parameter $\kappa$ assuming $\kappa\in[0,1]$. Under this condition, the derived distribution Eq. (7) *does reproduce* that of Haldane-Wu fractional exclusion statistics [Eq. (14) of Ref. [2]], with $\kappa=\lambda$.

Below, we turn to consideration of properties and exact solutions of Eq. (7).

In general, Eq. (7) cannot be solved exactly with respect to $n_i$. However, for $\kappa=1$ and $\kappa\to 0$ ($\kappa n_i\to 1$), it becomes linear in $n_i$ and gives Fermi and Bose distributions, respectively. Also, we note that for $\kappa=1/2$, 1/3, and 1/4 the equation contains a polynomial of degree up to 4 so that it can be solved exactly for all these cases.

A convenient expression for $n_i$ obeying Eq. (7) is given by [2]

$$n_i=\frac{1}{w(x)+\kappa}, \tag{8}$$

where we have redefined, $n_i/z_i\to n_i$, $x\equiv\exp[(\varepsilon_i-\mu)/kT]$, and the function $w(x)$ satisfies

$$(1+w)^{(1-\kappa)}w^{\kappa}=x. \tag{9}$$

Remarkably, exclusons that are "close" to fermions can be described in terms of exclusons that are "close" to bosons. In fact, we note that Eq. (7) is invariant under a set of transformations,

$$\kappa\to 1-\kappa,\quad n_i\to -n_i,\quad x\to -x \tag{10}$$

for $\kappa\neq 0,1$. Therefore, if $n_i(x,\kappa)$ satisfies Eq. (7) then the function $m_i=-n_i(-x,1-\kappa)$ satisfies the same equation. Thus, we obtain the following general relation:

$$n_i(-x,1-\kappa)=-n_i(x,\kappa),\qquad \kappa\neq 0,1. \tag{11}$$

We see that, e.g., the distribution $n_i$ of exclusons for $\kappa=1/200\simeq 0$ can be obtained from that of "dual" exclusons, with $\kappa=1-1/200=199/200\simeq 1$.

The values $\kappa=1$ and $\kappa\to 0$ ($\kappa n_i\to 1$) are the only two points of *degeneration* of Eq. (7). Hence, any "deviation" from Fermi or Bose statistics is characterized by a sharp change of statistical properties, sending us to consideration of exclusons. Consequently, we can divide particles into three main types, genuine fermions, genuine bosons, and exclusons ($\kappa\in]0,1[$), since their statistical distributions obey *different nondegenerate* equations.

A fixed point of the map $\kappa\to 1-\kappa$ is $\kappa=1/2$. Hence it represents a special case worth to be considered separately. In this case, Eq. (7) allows an exact solution and the result is (positive root) [2]

$$n_i=\frac{2}{\sqrt{1+4x^2}}=\frac{2}{\left(1+4\exp\frac{2(\varepsilon_i-\mu)}{kT}\right)^{1/2}}. \tag{12}$$

This distribution represents statistics with up to two particles in state, $q=2$ (semions).

We have obtained exact solutions (real roots) of Eq. (7) for $\kappa=1/3$ and 2/3 which we write as

$$n_i=\frac{3}{f+f^{-1}\mp 1}, \tag{13}$$

where

$$f=[2\sqrt{y(y\mp 1)}+2y\mp 1]^{1/3},\qquad y=2\left(\frac{3x}{2}\right)^3\pm 1. \tag{14}$$

From Eqs. (13) and (14) one can see how exclusons with $\kappa=1/3$ (upper sign) are related to exclusons with $\kappa=2/3$ (lower sign) that agrees with Eq. (11). Also, for $\kappa=1/4$ and 3/4 we have obtained the following exact solutions (positive real roots):

$$n_i=\frac{4}{\sqrt{2g^{-1/2}-g+3}\pm g^{1/2}\mp 2}, \tag{15}$$

where

$$g=\frac{3}{2}\{[z^2(z+2)]^{1/3}+[z(z+2)^2]^{1/3}\}+1, \tag{16}$$

$$z=\sqrt{3\left(\frac{4x}{3}\right)^4+1}-1. \tag{17}$$

Plots of $n_i(x)$ for various $\kappa$ are presented in Fig. 1, from which one can see that these exclusons behave similar to fermions.

Distributions of exclusons can be obtained from a different approach, based on the canonical statistical sum which implies the mean number of particles,

$$n=\frac{\sum_{N=0}^{q}Nx^{-N}}{\sum_{N=0}^{q}x^{-N}}. \tag{18}$$

This formula gives (exact) Fermi and Bose distributions for $q=1$ and $q\to\infty$, respectively, while for arbitrary $q\geqslant 1$ the sum is

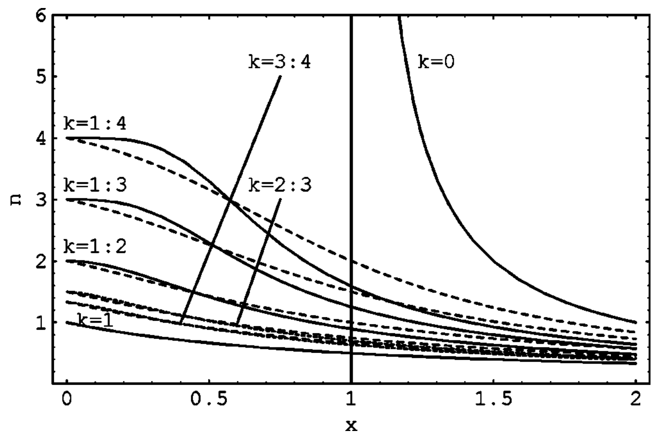


FIG. 1. Statistical distribution $n_i$ as a function of $x=\exp[(\varepsilon_i-\mu)/kT]$, for $\kappa=1$ (fermions), $\kappa=0$ (bosons), $\kappa=1/2$ [semions, Eq. (12)], $\kappa=1/3$ [Eq. (13), upper sign], $\kappa=1/4$ [Eq. (15), upper sign], $\kappa=2/3$ [Eq. (13), lower sign], and $\kappa=3/4$ [Eq. (15), lower sign]. Dashed lines represent the approximation (19) to exact solutions (solid lines).

$$n=\frac{x^{1+q}-(1+q)x+q}{(x^{1+q}-1)(x-1)}, \quad q=1/\kappa. \tag{19}$$

Distributions (19) are compared with exact solutions in Fig. 1. One can see that deviations become considerable as $\kappa$ goes to smaller values. However, we expect that near $\kappa=0$ there should be a better correspondence since one approaches the other interpolation end point (bosons). We treat Eq. (19) as an approximate result that is useful since it gives a single simple distribution formula for all exclusons, $\kappa\in[0,1]$.

A connection between the two approaches requires a deeper study which can be made elsewhere.

## IV. CONCLUSIONS

(i) The derived statistical weight (3) and Haldane-Wu statistical weight (1) lead to the same distribution Eq. (7);

(ii) Haldane-Wu parameter λ acquires a physical meaning of an inverse of the maximal allowed occupation number in state, $\lambda=1/q$, similar to the inverse of the statistical factor as shown by Wu [2];

(iii) Within fractional exclusion statistics, the generalized Pauli exclusion principle reads that a maximal allowed occupation number of identical particles in state is an integer, $q=1,2,3,\ldots$, i.e., $n_i/z_i\leqslant 1/\lambda$ as formulated by Wu [2]. We stress that in our approach we use this principle as a basis to calculate statistical weight (3) rather than derive it *a posteriori* from the analysis of a statistical weight or distribution function;

(iv) While Haldane-Wu parameter λ is assumed to vary continuously, the statistical parameter $\kappa=1/q$ runs over *discrete* set of values, $\kappa=1,1/2,1/3,\ldots$. This may be an important difference since physically acceptable solutions of Eq. (7) may not exist for all values of $\kappa\in]0,1[$, while $\kappa=1,1/2,1/3,\ldots$ guarantees a polynomial structure of Eq. (7), with physically acceptable solutions; and

(v) The Eq. (7), that defines statistical distribution of exclusons, $\kappa\in]0,1[$, has a remarkable symmetry (10) that allows to interconnect solutions $n_i$ for $\kappa$ and $1-\kappa$ due to Eq. (11).

In summary, we have shown that Haldane-Wu fractional exclusion statistics finds a natural combinatorial and physical interpretation in accord to Eq. (3), and presented exact solutions of Eq. (7).

[1] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).
[2] Y. S. Wu, Phys. Rev. Lett. **73**, 922 (1994).
[3] D. Bernard and Y. S. Wu, in *Proceedings of the 6th Nankai Workshop*, edited by M. L. Ge and Y. S. Wu (World Scientific, Singapore, 1995); C. Nayak and F. Wilczek, Phys. Rev. Lett. **73**, 2740 (1994); Z. N. C. Ha, Nucl. Phys. B **435**, 604 (1995); Y. Hatsugai, M. Kohmoto, T. Koma, and Y. S. Wu, Phys. Rev. B **54**, 5358 (1996); Y. Yu, H. X. Yang, and Y. S. Wu, e-print cond-mat/9911141; Y. S. Wu, Y. Yu, and H. X. Yang, Nucl. Phys. B **604**, 551 (2001).
[4] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).